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HARMONICITY AND MINIMALITY OF ORIENTED DISTRIBUTIONS

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ABSTRACT

We consider an oriented distribution as a section of the corresponding Grassmann bundle and, by computing the tension of this map for conveniently chosen metrics, we obtain the conditions which the distribution must satisfy in order to be critical for the functionals related to the volume or the energy of the map. We show that the three-dimensional distribution of S^{4m+3} tangent to the quaternionic Hopf fibration defines a harmonic map and a minimal immersion and we extend these results to more general situations coming from 3-Sasakian and quaternionic geometry.

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1. Introduction

Given any tensor bundle $\pi\colon P \to M$ and a Riemannian metric g_0 on M, the manifold P can be endowed, in a natural way, with a Riemannian metric which generalizes the Sasaki metric of the tangent bundle and which will be denoted by g_0^S. In the sequel, any smooth section σ of this bundle will be considered as a map from M into the Riemannian manifold (P, g_0^S); in fact, a section is a one-to-one immersion. The volume of σ will be the volume of the Riemannian manifold $(M, \sigma^* g_0^S)$ where $\sigma^* g_0^S$ is the metric induced by g_0^S. If $\tilde{g}$ is another metric on M, for any smooth section σ we can define the second fundamental form, the tension and the energy of σ as the corresponding items of the map $\sigma\colon (M, \tilde{g}) \to (P, g_0^S)$.

Our first aim is to obtain the expressions of these tensor fields. Moreover, since harmonic maps are characterized by the vanishing of the tension, we use the expression of these tensor fields, and that of some conveniently chosen projections of them, to write down not only the condition for a section to provide a harmonic map but also the condition for a unit section to provide a harmonic map into the unit bundle UP and the condition for a section to be a critical point of the energy for variations through sections; the latter will be called $\tilde{g}$-harmonic sections and the characterization was obtained in [27] by a different method (see also [22]). Two particular cases are important: when $\tilde{g} = g_0$, only a metric on M is involved and critical points are known as harmonic sections; when $\tilde{g} = \sigma^* g_0^S$, σ is a harmonic map if and only if σ is a $(\sigma^* g_0^S)$-harmonic section, or equivalently, if and only if σ defines a minimal immersion into the bundle. These results, which appeared in [11] for the particular case of vector fields, form the content of Section 2.

The third section is devoted to the study of oriented distributions. A q-dimensional oriented distribution on M is seen as a section of the Grassmann fiber bundle of q-planes and this one is viewed as a subbundle of the tensor bundle of q-vectors of M. This is the same idea as that used in [9] where the authors have obtained a condition for a distribution to be a harmonic section of the Grassmann bundle and used it to show that the quaternionic Hopf three-dimensional distribution of the sphere S^{4m+3} is harmonic. The same conclusion can also be obtained as a consequence of the result in [26] proving that the almost product structure associated to a Riemannian foliation with totally geodesic fibers of an Einstein manifold should be harmonic. We want to point out that, although the Grassmann fiber bundle is a homogeneous bundle, we do not use here this structure; recently, the energy of sections of homogeneous bundles has been studied in [28] and, in [10], an expression is obtained for the tension of a distribution, and hence a condition for it to be a harmonic map from (M, g_0) to the Grassmann bundle with the Sasaki metric g_0^S.

In this paper, our systematic approach allows us to go further and, by projecting the tension field obtained in Section 2, to obtain the condition for a distribution to be a harmonic map from $(M, \tilde{g})$ to the Grassmann bundle with the Sasaki metric g_0^S and, consequently, the condition for a distribution to be a minimal immersion. *When applying these results to the particular case of Hopf quaternionic distributions, we get that it is minimal and also that it defines a harmonic map when for $\tilde{g}$ we take any metric of the canonical variation of the submersion given by the Hopf fibration $\pi\colon S^{4m+3} \to \mathbb{H}P^m$.*

The three-dimensional quaternionic Hopf distribution is just the characteristic distribution of the usual 3-Sasakian structure on S^{4m+3}. The previously mentioned result can be generalized in a natural way to this situation and we obtain that for *any 3-contact metric manifold, the three-dimensional characteristic distribution is minimal and defines a harmonic map.*

Furthermore, we show that the energy and the volume of a distribution coincide with the energy and the volume of the orthogonal complementary distribution, and also that one of them is a harmonic map, $\tilde{g}$-harmonic or minimal, respectively, if and only if the other has the same property. As a consequence, the many known examples of unit vector fields (see Remark 3.5) give examples of codimension one distributions of the same kind.

If we regard the Hopf distribution on S^{4m+3} as the one generated by the product of the normal to the sphere by the three unit imaginary quaternions, that is to say, as the image of the normal vector field by the quaternionic distribution of $\mathbb{R}^{4m+4}$, it is natural to ask to what extend the results above can be generalized to the Hopf distribution of any oriented real hypersurface of a quaternionic Kähler manifold.

Following the lines developed in [23] and [24], where the authors studied the harmonicity and minimality of Hopf vector field on hypersurfaces of a Kähler manifold, we obtain in Section 4 the conditions that Hopf distributions of a hypersurface of a quaternionic Kähler manifold must satisfy in order to provide a harmonic map, a minimal immersion or a harmonic section. The conditions are rather complicated for a general hypersurface but, since the shape operator of the submanifold plays a central role, they simplify a great deal when we restrict to the well-known class of Hopf hypersurfaces, that is, those for which the shape operator preserves the Hopf distribution. This distribution then defines a totally geodesic foliation on the hypersurface. The conditions are written down in Theorem 4.5 and several applications are given, in particular when the ambient space is a quaternionic space form or the complex Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ and its non-compact dual. In this way we provide several examples of the considered special distributions. Other examples are given in [21].

2. Sections of tensor bundles that are harmonic maps

By a tensor bundle $\pi\colon P \to M$ we mean a vector bundle such that the fiber at each point P_x is a vector space consisting of tensors of the tangent space T_xM. That is, $P_x = (T_xM)_{(r,s)}$ for some $(r,s) \in \mathbb{Z} \times \mathbb{Z}$. In that case, the only data needed to construct a Riemannian metric on P is a Riemannian metric g_0 on M. This metric generalizes the Sasaki metric of the tangent bundle (see [5], for example). It will be denoted by g_0^S and, for $\xi_1, \xi_2 \in T_\sigma P$, it is defined by

$$g_0^S(\xi_1, \xi_2) = g_0(\pi_*(\xi_1), \pi_*(\xi_2)) + g_0(K(\xi_1), K(\xi_2)),$$

where $K\colon TP \to P$ is the connection map of the connection on P induced by the Levi Civita connection of g_0; we have represented by g_0 not only the metric on M but also the fibre metric induced on each fibre.

We recall that an element $\xi \in T_\sigma P$ is said to be **vertical** if $\pi_*(\xi) = 0$ and it is said to be **horizontal** if $K(\xi) = 0$.

To each smooth section σ of $\pi\colon P \to M$ we can associate a vertical vector field on P, denoted by σ^{vert}, and for each vector field X on M we can define a vector field X^{hor} in P, known as its horizontal lift. In a similar way as was done in [20] for the tangent bundle, we obtain for the Lie brackets the following expressions, for $X, Y \in \mathfrak{X}(M)$ and $\sigma, \eta \in \Gamma^\infty(P)$:

$$(2.1) \qquad \begin{cases} [X^{hor}, Y^{hor}] \circ \sigma = [X,Y]^{hor} \circ \sigma + (R(X,Y)\sigma)^{vert} \circ \sigma, \\ [X^{hor}, \sigma^{vert}] = (\nabla_X \sigma)^{vert}, \\ [\sigma^{vert}, \eta^{vert}] = 0. \end{cases}$$

Here $R(X,Y)\sigma = \nabla_{[X,Y]}\sigma - \nabla_X\nabla_Y\sigma + \nabla_Y\nabla_X\sigma$ and then, if we put

$$(\nabla^2\sigma)(X,Y) = \nabla_X\nabla_Y\sigma - \nabla_{(\nabla_X Y)}\sigma,$$

we have

$$R(X,Y)\sigma = (\nabla^2\sigma)(Y,X) - (\nabla^2\sigma)(X,Y).$$

Since any $\sigma \in \Gamma^\infty(P)$ is a map $\sigma\colon M \to (P, g_0^S)$, we now consider the tension of this map when M is equipped with a metric $\tilde{g}$ which, in view of our applications, may be different from g_0. It is the vector field on P along σ defined by

$$\tau_{\tilde{g}}(\sigma) = \sum_{i=1}^{n} \alpha_\sigma^{\tilde{g}}(\tilde{E}_i, \tilde{E}_i),$$

where $\{\tilde{E}_i\}_{i=1}^n$ is a local $\tilde{g}$-orthonormal frame and

$$\alpha_\sigma^{\tilde{g}}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma^\infty(\sigma^*TP)$$

is defined by $\alpha_\sigma^{\tilde{g}}(X,Y) = \nabla_X^S(\sigma_* \circ Y) - \sigma_* \circ \tilde{\nabla}_X Y$, where ∇^S is the Levi Civita connection of g_0^S. Now, we prove the following

THEOREM 2.1: *Given a section $\sigma \in \Gamma^\infty(P)$ of a tensor bundle $\pi\colon P \to M$ over a Riemannian manifold (M, g_0) and a metric $\tilde{g}$ on M, the second fundamental form and the tension of the map $\sigma\colon (M,\tilde{g}) \to (P, g_0^S)$ are given, respectively, by*

$$\begin{aligned} \alpha_\sigma^{\tilde{g}}(X,Y) =& (\nabla_X Y - \tilde{\nabla}_X Y)^{hor} \circ \sigma + \frac{1}{2}\sum_{i=1}^{n} g_0(R(E_i,Y)\sigma, \nabla_X\sigma)(E_i^{hor} \circ \sigma) \\ &+ \frac{1}{2}\sum_{i=1}^{n} g_0(R(E_i,X)\sigma, \nabla_Y\sigma)(E_i^{hor} \circ \sigma) \\ &+ \frac{1}{2}(R(X,Y)\sigma)^{vert} \circ \sigma + (\nabla_X\nabla_Y\sigma - \nabla_{\tilde{\nabla}_X Y}\sigma)^{vert} \circ \sigma \end{aligned}$$

and

$$\tau_{\tilde{g}}(\sigma) = (X_{(\sigma,\tilde{g})})^{hor} \circ \sigma + (\eta_{(\sigma,\tilde{g})})^{vert} \circ \sigma,$$

with

$$X_{(\sigma,\tilde{g})} = \sum_{i,j=1}^{n} g_0(R(E_i,\tilde{E}_j)\sigma, \nabla_{\tilde{E}_j}\sigma)E_i + \tau_{\tilde{g}}(\mathrm{Id})$$

and

$$\begin{aligned} \eta_{(\sigma,\tilde{g})} &= \nabla_{\tau_{\tilde{g}}(\mathrm{Id})}\sigma + \sum_{i=1}^{n}(\nabla^2\sigma)(\tilde{E}_i,\tilde{E}_i) \\ &= \sum_{i=1}^{n}(\nabla_{\tilde{E}_i}\nabla_{\tilde{E}_i}\sigma - \nabla_{(\tilde{\nabla}_{\tilde{E}_i}\tilde{E}_i)}\sigma) \end{aligned} \tag{2.1}$$

where $\tau_{\tilde{g}}(\mathrm{Id})$ is the tension of the identity map, considered from the Riemannian manifold $(M,\tilde{g})$ to (M,g_0), and $\{E_i\}_{i=1}^n$ and $\{\tilde{E}_i\}_{i=1}^n$ are local orthonormal frames with respect to g_0 and $\tilde{g}$, respectively.

Proof: Using (2.1) and Koszul's formula, we get for the Levi Civita connection of g_0^S:

$$\text{(2.3)} \qquad \begin{cases} \nabla^S_{\sigma^{vert}}\eta^{vert} = 0, \\ (\nabla^S_{X^{hor}}Y^{hor})\circ\sigma = (\nabla_X Y)^{hor}\circ\sigma + \frac{1}{2}(R(X,Y)\sigma)^{vert}\circ\sigma, \\ (\nabla^S_{\eta^{vert}}Y^{hor})\circ\sigma = \frac{1}{2}\sum_{i=1}^n g_0(R(E_i,Y)\sigma,\eta)(E_i^{hor}\circ\sigma). \end{cases}$$

These formulas are a special case of those derived in [5, Chapter 9] for vector bundles. To have the complete expression of ∇^S, we only need to have in mind that

$$\nabla^S_{Y^{hor}}\eta^{vert} = \nabla^S_{\eta^{vert}}Y^{hor} + [Y^{hor},\eta^{vert}].$$

On the other hand,

$$\text{(2.4)} \qquad \sigma_*\circ X = X^{hor}\circ\sigma + (\nabla_X\sigma)^{vert}\circ\sigma$$

and therefore

(2.5)

$$\begin{aligned} \nabla^S_X(\sigma_*\circ Y) =& (\nabla_X Y)^{hor}\circ\sigma + \frac{1}{2}(R(X,Y)\sigma)^{vert}\circ\sigma \\ &+ \frac{1}{2}\sum_{i=1}^n g_0(R(E_i,Y)\sigma,\nabla_X\sigma)(E_i^{hor}\circ\sigma) \\ &+ \frac{1}{2}\sum_{i=1}^n g_0(R(E_i,X)\sigma,\nabla_Y\sigma)(E_i^{hor}\circ\sigma) + (\nabla_X\nabla_Y\sigma)^{vert}\circ\sigma. \end{aligned}$$

Since

$$\alpha_\sigma^{\tilde g}(X,Y) = \nabla^S_X(\sigma_*\circ Y) - (\tilde\nabla_X Y)^{hor}\circ\sigma - (\nabla_{\tilde\nabla_X Y}\sigma)^{vert}\circ\sigma,$$

we get the result. ∎

In [19], Konderak has computed the tension of a section $\sigma\colon (M,g_0)\to(E,g^S)$, where E is any vector bundle over M and g^S is the Sasaki metric constructed with the metric g_0 on M, any metric h on the fiber and any connection D on the bundle. The particular case $\tilde g = g_0$ of the Theorem above is also a particular case of his result: for E a bundle of tensors on M, considered with the natural extensions of the metric g_0 and of the corresponding Levi Civita connection.

The results of Theorem 2.1 when P is the tangent bundle appeared in [11]. As was done there, it will be useful to write $\eta_{(\sigma,\tilde g)}$ in terms of the Levi Civita connection of g_0 and the endomorphism field $L_{\tilde g} = (\det(g_0^{-1}\tilde g))^{-\frac{1}{2}}(g_0^{-1}\tilde g)$, i.e., $L_{\tilde g} = (\det A)^{-\frac{1}{2}}A$ where $\tilde g(X,Y) = g_0(AX,Y)$ for all $X,Y\in\mathfrak{X}(M)$.

PROPOSITION 2.2: *Given a section $\sigma \in \Gamma^\infty(P)$ of a tensor bundle $\pi\colon P \to M$ over a Riemannian manifold (M, g_0) and a metric $\tilde{g}$ on M, denote by $K_{(\sigma,\tilde{g})}$ the section of $P \otimes T^*M$ given by $K_{(\sigma,\tilde{g})} = \nabla\sigma \circ L_{\tilde{g}}^{-1}$. Then*

$$\eta_{(\sigma,\tilde{g})} = -(\det(g_0^{-1}\tilde{g}))^{-\frac{1}{2}}\nabla^*(K_{(\sigma,\tilde{g})}),$$

where $\nabla^*\colon \Gamma^\infty(P \otimes T^*M) \to \Gamma^\infty(P)$ represents the divergence defined as

$$\nabla^*(K) = -\sum_{i=1}^{n}(\nabla_{E_i}K)E_i.$$

Proof: It has been shown in [11] that

$$\sum_{i=1}^{n}\tilde{\nabla}_{\tilde{E}_i}\tilde{E}_i = \sum_{i=1}^{n}L_{\tilde{g}}^{-1}(\nabla_{\tilde{E}_i}L_{\tilde{g}}(\tilde{E}_i)).$$

Using now (2.2), we get

$$\eta_{(\sigma,\tilde{g})} = \sum_{i=1}^{n}(\nabla_{\tilde{E}_i}(\nabla\sigma \circ L_{\tilde{g}}^{-1}))(L_{\tilde{g}}(\tilde{E}_i)).$$

By the choice of $L_{\tilde{g}}$, we have for $X \in \mathfrak{X}(M)$:

$$X = \sum_{i=1}^{n} g_0(X, E_i)E_i = (\det(g_0^{-1}\tilde{g}))^{-\frac{1}{2}}\sum_{i=1}^{n}\tilde{g}(L_{\tilde{g}}^{-1}(X), E_i)E_i,$$

from where the result follows. ∎

COROLLARY 2.3: *Given a section $\sigma \in \Gamma^\infty(P)$ of a tensor bundle $\pi\colon P \to M$ over a Riemannian manifold (M, g_0) and a metric $\tilde{g}$ on M, the map $\sigma\colon (M, \tilde{g}) \to (P, g_0^S)$ is harmonic if and only if $X_{(\sigma,\tilde{g})} = 0$ and $\eta_{(\sigma,\tilde{g})} = 0$. If σ is a section of the subbundle UP consisting of those tensors of unit norm, then the map $\sigma\colon (M, \tilde{g}) \to (UP, g_0^S)$ is harmonic if and only if $X_{(\sigma,\tilde{g})} = 0$ and $\eta_{(\sigma,\tilde{g})}$ is collinear to σ.*

Proof: The first assertion comes from the theorem and the definition of harmonic maps as those with vanishing tension. Furthermore, a unit σ is harmonic as a map to (UP, g_0^S) if and only if for all $p \in M$ the vector $\tau_{\tilde{g}}(\sigma)(p) \in T_{\sigma(p)}P$ is collinear to the vector field normal to the immersion $UP \subset P$ which is given by $\sigma^{vert}(\sigma(p))$, from where the result follows. ∎

The energy of the map $\sigma\colon (M, \tilde{g}) \to (P, g_0^S)$ is

$$E(\sigma, \tilde{g}) = \frac{1}{2}\int_M \operatorname{trace}(L_{(\sigma,\tilde{g})})dv_{\tilde{g}},$$

where $L_{(\sigma,\tilde{g})}$ is the endomorphism field uniquely determined by

$$(\sigma^* g_0^S)(X,Y) = \tilde{g}(L_{(\sigma,\tilde{g})}X, Y).$$

In the particular case $\tilde{g} = g_0$, $E(\sigma, g_0) = E(\sigma)$ is called the **energy of the section** and $E(\sigma, \sigma^* g_0^S) = \frac{1}{2} n \operatorname{vol}(\sigma)$, where $\operatorname{vol}(\sigma)$ is the n-dimensional volume of the submanifold $\sigma(M)$, or equivalently, the volume of the Riemannian manifold $(M, \sigma^* g_0^S)$.

If M is compact, σ is a **harmonic map** if and only if it is a critical point of the energy functional from $C^\infty(M,P)$ to $\mathbb{R}$ and σ is a **minimal immersion** if and only if it is a critical point of the volume functional from $\operatorname{Imm}(M,P)$ to $\mathbb{R}$.

Once we know the condition for a smooth section of P to provide a harmonic map, we can consider the restriction of the energy functional to the space of sections. Let $E_{\tilde{g}}$ be the functional that maps each smooth section σ into $E(\sigma, \tilde{g})$. A critical point $\sigma \in \Gamma^\infty(P)$ of $E_{\tilde{g}}$ is called a **$\tilde{g}$-harmonic section** or simply **harmonic** if $\tilde{g} = g_0$. Since the variational field $\sigma'(t)$ of any variation $\sigma(t)$ of σ through sections of $\pi\colon P \to M$ is in the vertical subbundle $\ker \pi_* \subset TP$, critical points of this restricted functional are characterized by the vanishing of the vertical component of their tension and then we have

COROLLARY 2.4: *A section $\sigma \in \Gamma^\infty(P)$ is $\tilde{g}$-harmonic if and only if $\eta_{(\sigma,\tilde{g})} = 0$ and a unit section is a critical point of $E_{\tilde{g}}$ restricted to $\Gamma^\infty(UP)$ if and only if $\eta_{(\sigma,\tilde{g})}$ is collinear to σ.*

PROPOSITION 2.5: *Given a section $\sigma \in \Gamma^\infty(P)$ of a tensor bundle $\pi\colon P \to M$ over a Riemannian manifold (M, g_0), the immersion $\sigma\colon M \to (P, g_0^S)$ is minimal if and only if $\eta_{(\sigma,\tilde{g})} = 0$, where $\tilde{g} = \sigma^* g_0^S$. If σ is a unit section, then it is a minimal immersion into (UP, g_0^S) if and only if $\eta_{(\sigma,\tilde{g})}$ is collinear to σ.*

Proof: If we compare the tension of a map with the mean curvature vector field of an immersion, it is clear that the immersion σ is minimal if and only if it is a harmonic map from $(M, \tilde{g})$, with $\tilde{g} = \sigma^* g_0^S$.

In view of Corollary 2.3, we only need to show that, for this particular value of $\tilde{g}$, if $\eta_{(\sigma,\tilde{g})} = 0$ (or if σ is a unit section and $\eta_{(\sigma,\tilde{g})}$ is collinear to σ), then $X_{(\sigma,\tilde{g})} = 0$. This will be a consequence of the following equality, for all $X \in \mathfrak{X}(M)$:

$$g_0(X_{(\sigma,\sigma^* g_0^S)}, X) + g_0(\eta_{(\sigma,\sigma^* g_0^S)}, \nabla_X \sigma) = 0. \tag{2.6}$$

In fact, if a vector field $\xi \in \mathfrak{X}(P)$ is of the form $\xi = Y^{hor} + \eta^{vert}$ and such that $\xi \circ \sigma$ is orthogonal to the submanifold $\sigma(M)$, with respect to the metric

g_0^S, then for all $X \in \mathfrak{X}(M)$ we have, using (2.4) and the definition of g_0^S, that

$$g_0(Y, X) + g_0(\eta, \nabla_X \sigma) = g_0^S(\xi \circ \sigma, \sigma_* \circ X) = 0.$$

Since $\tau_{\tilde{g}}(\sigma)$ is the mean curvature vector field of the immersion, it is orthogonal to $\sigma(M)$, from where (2.6) holds.

3. Energy and volume of oriented distributions

Let (M, g_0) be a Riemannian manifold and let

$$\Lambda^q(M) = \bigcup_{x \in M} \Lambda^q(T_x M)$$

be the tensor bundle of all skew-symmetric contravariant tensors of order q, or briefly, of all q-vectors on M. Now, we consider the Grassmann bundle as a subbundle of $\Lambda^q(M)$. Let V be an n-dimensional oriented vector space and let $G_q^o(V)$ be the oriented Grassmann manifold, i.e., the set of all oriented q-dimensional linear subspaces of V. Then $G_q^o(V)$ is an $(n-q)q$-dimensional compact homogeneous Riemannian manifold. Moreover, if we choose an inner product on V, we may identify $G_q^o(V)$ in a natural way with the set

$$\{\sigma \in \Sigma_q(V); \|\sigma\| = 1\}$$

where $\Sigma_q(V)$ denotes the set of all decomposable q-vectors of V. In fact, each q-dimensional subspace U of V is identified with the decomposable q-vector $\sigma = e_1 \wedge \cdots \wedge e_q$ of $\Lambda^q(V)$ where $\{e_1, \ldots, e_q\}$ is a positive orthonormal basis of U. We shall say that U is the **associated subspace** of σ. It can be shown that this representation of $G_q^o(V)$ is a submanifold of $\Lambda^q(V)$. Furthermore, the tangent space of $G_q^o(V)$ at a subspace U represented by $\sigma = e_1 \wedge \cdots \wedge e_q$ corresponds to the subspace $T_\sigma G_q^o(V) \subset \Lambda_q(V)$ generated by

$$\{\sigma_j^a = e_1 \wedge \cdots e_{a-1} \wedge e_{j+q} \wedge e_{a+1} \cdots \wedge e_q;\ a = 1, \ldots, q,\ j = 1, \ldots, n-q\}$$

where $\{e_{q+1}, \ldots, e_n\}$ are chosen so that they complete the positive orthonormal basis $\{e_1, \ldots, e_q\}$ of U to a positive orthonormal basis of V.

In fact, let $\sigma(t) = e_1(t) \wedge \cdots \wedge e_q(t)$ be a differentiable curve in $\Lambda^q(V)$, lying in $G_q^o(V)$, with $\sigma(0) = \sigma$. Then

$$\sigma'(0) = \sum_{a=1}^{q} e_1 \wedge \cdots \wedge e_a'(0) \wedge \cdots \wedge e_q.$$

Since $e_a(t)$ is a unit vector, $e'_a(0)$ is orthogonal to e_a and so $\sigma'(0)$ is in the subspace generated by $\{\sigma^a_j\}$.

Let $G^o_q(M) = \bigcup_{x\in M} G^o_q(T_xM)$ be the **Grassmann fibre bundle**. Then, under the above identification, $G^o_q(M)$ is the unit decomposable subbundle of $\Lambda^q(M)$ and each q-dimensional oriented and smooth distribution $\mathcal{V}$ on M gives a section $\sigma \in \Gamma^\infty(G^o_q(M))$ of the Grassmann bundle and may be considered as a global smooth section of the tensor bundle $\Lambda^q(M)$, also denoted by σ. It can be expressed locally as $\sigma = E_1 \wedge \cdots \wedge E_q$, where $\{E_1, \ldots, E_n\}$ is a positive orthonormal local frame such that $E_1, \ldots, E_q$ span $\mathcal{V}$ and $E_{q+1}, \ldots, E_n$ span its g_0-orthogonal complementary distribution $\mathcal{H}$. We will denote the corresponding section of $G^o_{n-q}(M)$ by $\sigma^\perp$. It is locally expressed as $\sigma^\perp = E_{q+1}\wedge\cdots\wedge E_n$. Such a local frame will be called **a local frame adapted to the distribution.**

The energy of the distribution $\mathcal{V}$ is then defined as the energy of the corresponding section σ, where $G^o_q(M)$ is considered with the induced Sasaki metric from $\Lambda^q(M)$.

PROPOSITION 3.1: *Let σ be a q-dimensional oriented distribution on a Riemannian manifold (M, g_0) and let $\sigma^\perp$ be the corresponding g_0-orthogonal distribution. Then $\sigma^* g_0^S = (\sigma^\perp)^* g_0^S$ and consequently, for any metric $\tilde{g}$ on M, the maps $\sigma\colon (M,\tilde{g}) \to (G^o_q(M), g_0^S)$ and $\sigma^\perp\colon (M,\tilde{g}) \to (G^o_{n-q}(M), g_0^S)$ have the same energy.*

Proof: Since $\sigma^* g_0^S(X,Y) = g_0(X,Y) + g_0(\nabla_X\sigma, \nabla_Y\sigma)$, we only need to show that

$$g_0(\nabla_X\sigma, \nabla_Y\sigma) = g_0(\nabla_X\sigma^\perp, \nabla_Y\sigma^\perp).$$

If we compute $\nabla_X\sigma$ in an adapted local frame, we obtain

$$\nabla_X\sigma = \sum_{a=1}^{q}\sum_{j=1}^{n-q} g_0(\nabla_X E_a, E_{q+j})\sigma^a_j, \tag{3.1}$$

and then

$$g_0(\nabla_X\sigma, \nabla_Y\sigma) = \sum_{a,b=1}^{q}\sum_{j,k=1}^{n-q} g_0(\nabla_X E_a, E_{q+j}) g_0(\nabla_Y E_b, E_{q+k}) g_0(\sigma^a_j, \sigma^b_k).$$

Since $g_0(\sigma_j^a, \sigma_k^b) = \delta_j^k \delta_a^b$,

$$\begin{aligned} g_0(\nabla_X \sigma, \nabla_Y \sigma) &= \sum_{a=1}^{q} \sum_{j=1}^{n-q} g_0(\nabla_X E_a, E_{q+j}) g_0(\nabla_Y E_a, E_{q+j}) \\ &= \sum_{a=1}^{q} \sum_{j=1}^{n-q} g_0(E_a, \nabla_X E_{q+j}) g_0(E_a, \nabla_Y E_{q+j}) \\ &= g_0(\nabla_X \sigma^\perp, \nabla_Y \sigma^\perp). \qquad \blacksquare \end{aligned}$$

If we denote by $S^0_{\sigma(x)}$ the subspace of $\Lambda^q(T_xM)$ generated by $\sigma(x)$, by $S^1_{\sigma(x)}$ the subspace $T_{\sigma(x)}G^o_q(T_xM)$ previously described, and by $S^2_{\sigma(x)}$ the subspace generated by

$$\{\sigma_{ij}^{ab}(x)\ a, b = 1, \ldots, q;\ i, j = 1, \ldots, n-q\},$$

where $\sigma_{ij}^{ab}(x) = e_1 \wedge \cdots \wedge e_{a-1} \wedge e_{q+i} \wedge e_{a+1} \wedge \cdots \wedge e_{b-1} \wedge e_{q+j} \wedge e_{b+1} \wedge \cdots \wedge e_q$, then it is easy to see that for every $\sigma \in \Gamma^\infty(G^o_q(M))$ we have

$$\eta_{(\sigma,\tilde{g})}(x) \in S^0_{\sigma(x)} \oplus S^1_{\sigma(x)} \oplus S^2_{\sigma(x)}.$$

The condition for σ to be harmonic as a map into $G^o_q(M)$ is the vanishing of the projection of the tension onto $T_{\sigma(x)}G^o_q(T_xM)$ and we have shown the following:

THEOREM 3.2: *Given an oriented q-dimensional distribution on a Riemannian manifold* (M, g_0) *and a metric* $\tilde{g}$ *on* M, *then*

(a) *the corresponding map* $\sigma\colon (M, \tilde{g}) \to (G^o_q(M), g_0^S)$ *is harmonic if and only if*

$$X_{(\sigma,\tilde{g})} = 0$$

and for all $x \in M$,

$$\eta_{(\sigma,\tilde{g})}(x) \in S^0_{\sigma(x)} \oplus S^2_{\sigma(x)}; \tag{3.2}$$

(b) *the condition* (3.2) *characterizes these* $\tilde{g}$*-harmonic distributions or, equivalently, those distributions that are critical points of* $E_{\tilde{g}}$ *restricted to* $\Gamma^\infty(G^o_q(M))$;

(c) *the immersion* $\sigma\colon M \to (G^o_q(M), g_0^S)$ *is minimal if and only if*

$$\eta_{(\sigma,\sigma^* g_0^S)}(x) \in S^0_{\sigma(x)} \oplus S^2_{\sigma(x)}.$$

Since the subspaces $S^0_{\sigma(x)}$, $S^1_{\sigma(x)}$ and $S^2_{\sigma(x)}$ are mutually orthogonal, (3.2) is equivalent to $g_0(\eta_{(\sigma,\tilde{g})}, \sigma_j^a) = 0$ for all $a = 1, \ldots, q$, $j = 1, \ldots, n-q$. Therefore, in view of our applications, the following lemma will be useful.

LEMMA 3.3: *Let σ be a q-dimensional distribution on a Riemannian manifold (M, g_0) and let P be an endomorphism field. If we put $K_\sigma = \nabla\sigma \circ P$, then*

$$g_0(R(X,Y)\sigma, \sigma_j^a) = g_0(R(X,Y)E_a, E_{q+j}) \tag{3.3}$$

and

$$\begin{aligned} g_0((\nabla_X K_\sigma)X, \sigma_j^a) =& g_0(\nabla_X \nabla_{PX} E_a - \nabla_{P(\nabla_X X)} E_a, E_{q+j}) \\ &+ \sum_{b=1}^{q} (g_0(\nabla_X E_a, E_b) g_0(\nabla_{PX} E_{q+j}, E_b) \\ &\quad + g_0(\nabla_{PX} E_a, E_b) g_0(\nabla_X E_{q+j}, E_b)). \end{aligned} \tag{3.4}$$

Proof: Since from (3.1) we get

$$\nabla_X \nabla_Y \sigma = \sum_{a=1}^{q} \sum_{j=1}^{n-q} \{X g_0(\nabla_Y E_a, E_{q+j}) \sigma_j^a + g_0(\nabla_Y E_a, E_{q+j}) \nabla_X \sigma_j^a\}$$

and the projection of $\nabla_X \sigma_j^a$ onto S_σ^1 is given by

$$\sum_{k=1}^{n-q} g_0(\nabla_X E_{q+j}, E_{q+k}) \sigma_k^a - \sum_{b=1}^{q} g_0(\nabla_X E_b, E_a) \sigma_j^b,$$

we have

$$\begin{aligned} g_0(\nabla_X \nabla_Y \sigma, \sigma_j^a) =& g_0(\nabla_X \nabla_Y E_a, E_{q+j}) + g_0(\nabla_Y E_a, \nabla_X E_{q+j}) \\ &+ \sum_{k=1}^{n-q} g_0(\nabla_Y E_a, E_{q+k}) g_0(\nabla_X E_{q+k}, E_{q+j}) \\ &- \sum_{b=1}^{q} g_0(\nabla_X E_a, E_b) g_0(\nabla_Y E_b, E_{q+j}). \end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned} g_0(\nabla^2_{XY} \sigma, \sigma_j^a) =& g_0(\nabla^2_{XY} E_a, E_{q+j}) \\ &+ \sum_{b=1}^{q} (g_0(\nabla_X E_a, E_b) g_0(\nabla_Y E_{q+j}, E_b) \\ &\quad + g_0(\nabla_Y E_a, E_b) g_0(\nabla_X E_{q+j}, E_b)) \end{aligned} \tag{3.6}$$

from where (3.3) follows. Using again (3.5) and taking into account that

$$(\nabla_X K_\sigma)X = \nabla_X \nabla_{PX} \sigma - \nabla_{P(\nabla_X X)} \sigma, \tag{3.7}$$

we obtain (3.4). ∎

PROPOSITION 3.4: *Let σ be a q-dimensional distribution on a Riemannian manifold (M, g_0) and let $\sigma^\perp$ be the corresponding g_0-orthogonal distribution. Then for any metric $\tilde{g}$ on M, the map $\sigma\colon (M,\tilde{g}) \to (G_q^o(M), g_0^S)$ is harmonic if and only if the map $\sigma^\perp\colon (M,\tilde{g}) \to (G_{n-q}^o(M), g_0^S)$ is harmonic, and the distribution σ is $\tilde{g}$-harmonic if and only if the distribution $\sigma^\perp$ is $\tilde{g}$-harmonic. Moreover, the immersion $\sigma\colon M \to (G_q^o(M), g_0^S)$ is minimal if and only if $\sigma^\perp\colon M \to (G_{n-q}^o(M), g_0^S)$ is minimal.*

Proof: Using (3.1) and (3.3), we get

$$(3.8) \qquad g_0(R(X,Y)\sigma, \nabla_Y \sigma) = \sum_{a=1}^{q}\sum_{j=1}^{n-q} g_0(\nabla_Y E_a, E_{q+j}) g_0(R(X,Y)E_a, E_{q+j}).$$

Hence, $X_{(\sigma,\tilde{g})} = X_{(\sigma^\perp,\tilde{g})}$. On the other hand, (3.4) yields

$$\begin{aligned} g_0((\nabla_X K_\sigma)X, \sigma_j^a) =& X g_0(\nabla_{PX} E_a, E_{q+j}) - g_0(\nabla_{P(\nabla_X X)} E_a, E_{q+j}) \\ &+ \sum_{k=1}^{n-q} g_0(\nabla_{PX} E_a, E_{q+k}) g_0(\nabla_X E_{q+k}, E_{q+j}) \\ &- \sum_{b=1}^{q} g_0(\nabla_X E_a, E_b) g_0(\nabla_{PX} E_b, E_{q+j}) \end{aligned}$$

for all $a \in \{1, \ldots, q\}$ and $j \in \{1, \ldots, n-q\}$. So, we have

$$g_0((\nabla_X K_\sigma)X, \sigma_j^a) = -g_0((\nabla_X K_{\sigma^\perp})X, (\sigma^\perp)_a^j)$$

and this implies, from Proposition 2.2, that

$$g_0(\eta_{(\sigma,\tilde{g})}, \sigma_j^a) = -g_0(\eta_{(\sigma^\perp,\tilde{g})}, (\sigma^\perp)_a^j).$$

Now, the result follows from Theorem 3.2. ∎

Remark 3.5: As a consequence of Proposition 3.1, any example in which the value of the energy or of the volume of a unit vector field has been computed, also gives the value of the energy or the volume of the orthogonal distribution. Moreover, every critical unit vector field provides a critical codimension one distribution as a consequence of Proposition 3.4. So, all the examples given in [3, 6, 7, 8, 12, 13, 14, 15, 16, 17, 23, 24, 25] also provide examples of codimension one minimal or harmonic distributions and of harmonic maps. In particular, we formulate one appearing in contact metric geometry.

It is well known that the characteristic vector field of a K-contact manifold (M,g,ξ) is an eigenvector of the Ricci operator and in [25] it has been shown that this is exactly the condition for a Killing vector field to be harmonic. That the characteristic vector field is also minimal is obtained in [14], by showing that the endomorphism field A relating the metric $\tilde{g}=\xi^*g^S$ with the metric g verifies $A(\xi)=\xi$ and $A|\xi^\perp = 2\mathrm{Id}$. Now, it has been seen in [11] that $X_{(\xi,\tilde{g})}=0$ for a minimal unit vector field. For this particular value of A, by using the formula for $X_{(\xi,\tilde{g})}$ given in Theorem 2.1 and by showing that the identity map is in this case harmonic (use the first formula in the proof of Proposition 2.2), this implies that $X_{(\xi,g)}=0$ too and hence that ξ determines a harmonic map. So we have

PROPOSITION 3.6: *The characteristic vector field and the corresponding characteristic distribution of a K-contact manifold are minimal and define harmonic maps.*

Next, we provide some other examples. If we represent by N the unit outward normal to the sphere S^n, $n=4m-1$, $m\geq 1$, in $\mathbb{R}^{4m}$ and by $\{I,J,K\}$ the usual quaternionic structure of $\mathbb{R}^{4m}$, then the 3-vector field σ globally defined by $\sigma = IN\wedge JN\wedge KN$ determines the smooth three-dimensional distribution $\mathcal{V}$ on S^n generated by $\{IN,JN,KN\}$ tangent to the Hopf fibration $\pi\colon S^n\to\mathbb{H}P^{m-1}$ and we will refer to it as the **Hopf distribution**. If g_0 is the standard metric on the sphere and $\mathcal{H}$ represents the distribution g_0-orthogonal to $\mathcal{V}$, we consider on S^n the canonical variation g_t, i.e., $g_t|_{\mathcal{V}}=g_0$, $g_t|_{\mathcal{H}}=tg_0$ and $g_t(\mathcal{V},\mathcal{H})=0$. For all $t>0$, $\pi\colon(S^n,g_t)\to\mathbb{H}P^{m-1}$ is a Riemannian submersion with totally geodesic fibres and then the map Id: $(S^n,g_t)\to(S^n,g_0)$ is harmonic. It is not difficult to show that $X_{(\sigma,g_t)}=0$ and that

$$\begin{aligned}\eta_{(\sigma,g_t)} ={}&\frac{3(3-n)}{t}\sigma\\ &+2\sum_{j=1}^{n-3}(I\tilde{E}_j\wedge J\tilde{E}_j\wedge KN+I\tilde{E}_j\wedge JN\wedge K\tilde{E}_j+IN\wedge J\tilde{E}_j\wedge K\tilde{E}_j),\end{aligned}$$

where $\{\tilde{E}_j\}_{j=1}^{n-3}$ is a local g_t-orthonormal frame of $\mathcal{H}$.

Hence, $\eta_{(\sigma,g_t)}(x)\in S^0_{\sigma(x)}\oplus S^2_{\sigma(x)}$ for all $x\in S^n$. From Corollary 2.3 we get that σ is not a harmonic map from (S^n,g_t) to $(\Lambda^3(S^n),g_0^S)$, but using the characterization of Theorem 3.2 we get

PROPOSITION 3.7: *Let $\tilde{g}$ be any of the metrics of the canonical variation of the Riemannian submersion $\pi\colon S^n\to\mathbb{H}P^{m-1}$. Then the three-dimensional Hopf distribution provides a harmonic map from $(S^n,\tilde{g})$ to $(G_3^o(S^n),g_0^S)$.*

Remark 3.8: As a consequence, the three-dimensional Hopf distribution is harmonic; this result also appeared in [9]. Moreover, it provides a harmonic map from (S^n, g_0) to $(G_3^o(S^n), g_0^S)$ which has been shown in [10] by a different method: using the homogeneous structure of the sphere and the structure of homogeneous bundle of $G_3^o(S^n)$.

PROPOSITION 3.9: *The three-dimensional Hopf distribution on S^n provides a minimal immersion of the sphere into its Grassmann bundle of oriented 3-planes.*

Proof: It is straightforward to show that the metric $\sigma^* g_0^S$ restricted to $\mathcal{V}$ is equal to g_0 and restricted to $\mathcal{H}$ it is equal to $4g_0$. Then it is homothetic to one of the metrics g_t and then the result follows from the above Proposition. ∎

PROPOSITION 3.10: *The three-dimensional characteristic distribution on any n-dimensional 3-contact metric manifold (M, g_0) provides a minimal immersion of the manifold into its Grassmann bundle of oriented 3-planes and also a harmonic map from (M, g_0). Moreover, if M has finite volume, $E(\sigma) = 2(n-3)\,\mathrm{vol}(M)$ and the volume $F(\sigma)$ of σ is given by $F(\sigma) = 2^{n-3}\,\mathrm{vol}(M)$.*

Proof: First, we recall that due to a recent result by Kashiwada [18], every 3-contact metric manifold is 3-Sasakian. We refer to [5] for more details and also for the needed results and formulas from Sasakian geometry. If we represent by ξ_a, $a = 1, 2, 3$, the unit vector fields defining the structure, then $\sigma = \xi_1 \wedge \xi_2 \wedge \xi_3$ and an adapted local frame $\{E_\alpha\}_{\alpha=1}^n$ can be chosen such that $E_a = \xi_a$, $a = 1, 2, 3$. Since on a Sasakian manifold

$$R(X,Y)\xi_a = g_0(X,\xi_a)Y - g_0(Y,\xi_a)X$$

for all vector fields X, Y, (3.8) implies

$$\begin{aligned} g_0(R(X,Y)\sigma, \nabla_Y\sigma) = &\sum_{a=1}^{3}\sum_{j=1}^{n-3} g_0(\nabla_Y\xi_a, E_{j+3})g_0(X,\xi_a)g_0(Y,E_{j+3}) \\ &-\sum_{a=1}^{3}\sum_{j=1}^{n-3} g_0(\nabla_Y\xi_a, E_{j+3})g_0(Y,\xi_a)g_0(X,E_{j+3}). \end{aligned}$$

Each ξ_a is Killing and consequently, $X_{(\sigma,g_0)} = 0$. Furthermore, we show that $g_0(\eta_{(\sigma,g_0)}, \sigma_j^a) = 0$. By (3.4) we have

$$g_0((\nabla_X\nabla\sigma)X, \sigma_j^a) = g_0((\nabla_X\nabla\xi_a)X, E_{3+j}) + 2\sum_{b=1}^{3} g_0(\nabla_X\xi_a, \xi_b)g_0(\nabla_X E_{3+j}, \xi_b).$$

We also have $\nabla\xi_a = -\varphi_a$ and $(\nabla_X\varphi_a)X = g_0(X,X)\xi_a - g_0(\xi_a,X)X$. As a consequence,

$$\begin{aligned} g_0((\nabla_{E_\alpha}\nabla\sigma)E_\alpha,\sigma_j^a) &\\ &= g_0(\xi_a,E_\alpha)g_0(E_\alpha,E_{3+j}) + 2\sum_{b=1}^{3} g_0(\nabla_{E_\alpha}\xi_b,\xi_a)g_0(\nabla_{E_\alpha}\xi_b,E_{3+j})\\ &= 2\sum_{b=1}^{3} g_0(\nabla_{\xi_a}\xi_b,E_\alpha)g_0(\nabla_{E_{3+j}}\xi_b,E_\alpha)\end{aligned}$$

and then

$$\begin{aligned} g_0(\eta_{(\sigma,g_0)},\sigma_j^a) &= -2\sum_{b=1}^{3} g_0(\nabla_{\xi_a}\xi_b,\nabla_{E_{3+j}}\xi_b)\\ &= -2\sum_{b=1}^{3} g_0(\varphi_b\xi_a,\varphi_b E_{3+j}) = 0.\end{aligned}$$

Now, we compute $\sigma^* g_0^S$. From (3.1) and the definition of g_0^S, we have

$$\begin{aligned}\sigma^* g_0^S(X,Y) - g_0(X,Y) &= \sum_{a=1}^{3}\sum_{j=1}^{n-3} g_0(\varphi_a X,E_{3+j})g_0(\varphi_a Y,E_{3+j})\\ &= \sum_{a=1}^{3}\left\{g_0(\varphi_a X,\varphi_a Y) - \sum_{b=1}^{3} g_0(\varphi_a X,\xi_b)g_0(\varphi_a Y,\xi_b)\right\}.\end{aligned}$$

Since $g_0(\varphi_a X,\varphi_a Y) = g_0(X,Y) - g_0(X,\xi_a)g_0(Y,\xi_a)$, we obtain

$$\sigma^* g_0^S(X,Y) = 4g_0(X,Y) - 3\sum_{a=1}^{3} g_0(X,\xi_a)g_0(Y,\xi_a).$$

From here, and if we denote by A the endomorphism field such that $\sigma^* g_0^S(X,Y) = g_0(AX,Y)$, we have $A(\xi_a) = \xi_a$ and $A(E_{3+j}) = 4E_{3+j}$ from where we obtain the values given in the statement for the volume and the energy of σ and also that, for $\tilde g = \sigma^* g_0^S$, $L_{\tilde g} = 2^{3-n}A$. Using now (3.4) with $P = L_{\tilde g}^{-1}$, it is straightforward to show that

$$g_0(\eta_{(\sigma,\tilde g)},\sigma_j^a) = 2^{n-5} g_0(\eta_{(\sigma,g_0)},\sigma_j^a)$$

and so σ is also minimal. ∎

Remark 3.11: It follows from Proposition 3.4 that the same results are true for the distribution $\sigma^\perp$ which in that case has dimension $n-3=4m$.

4. Quaternionic Hopf distributions

In this section, we shall put the examples given in Section 3 in a broader context by considering Hopf distributions on real hypersurfaces of quaternionic Kähler manifolds of dimension $4m$ with $m > 1$. Before starting the study of the critical points of the energy and volume functionals, we start by recalling some general preliminaries about quaternionic Kähler manifolds. For more details, we refer to [1] and [4].

A **quaternionic Kähler structure** on a Riemannian manifold $(\bar{M}, g)$ is a rank 3 vector subbundle $\bar{\mathcal{J}}$ of $\mathrm{End}(T\bar{M})$ with the following properties:

(a) For each $p \in \bar{M}$, there exists an open neighbourhood $\bar{\mathcal{U}}$ of p in $\bar{M}$ and sections J_1, J_2, J_3 of $\bar{\mathcal{J}}$ over $\bar{\mathcal{U}}$ such that for all $a \in \{1,2,3\}$:

(1) J_a is an almost Hermitian structure on $\bar{\mathcal{U}}$, that is, $J_a^2 = -\mathrm{Id}_{|\bar{\mathcal{U}}}$ and

$$g(J_aX, Y) + g(X, J_aY) = 0 \quad \text{for all } X, Y \in \mathfrak{X}(\bar{\mathcal{U}});$$

(2) $J_aJ_{a+1} = J_{a+2} = -J_{a+1}J_a \ (a \bmod 3)$.

(b) $\bar{\mathcal{J}}$ is a parallel subbundle of $\mathrm{End}(T\bar{M})$, that is, $\bar{\nabla}_X J \in \Gamma(\bar{\mathcal{J}})$ for all $X \in \mathfrak{X}(\bar{M})$, $J \in \Gamma(\bar{\mathcal{J}})$.

A triple $\{J_1, J_2, J_3\}$ satisfying these conditions is called a **canonical local basis** of $\bar{\mathcal{J}}$. From (b) it follows that there exist three one-forms q_1, q_2, q_3 on $\bar{\mathcal{U}}$ such that

$$\bar{\nabla}_X J_a = q_{a+2}(X)J_{a+1} - q_{a+1}(X)J_{a+2} \ (a \bmod 3) \tag{4.1}$$

for all $X \in \mathfrak{X}(\bar{\mathcal{U}})$, $a \in \{1,2,3\}$.

A Riemannian manifold $(\bar{M}, g)$ equipped with a quaternionic Kähler structure is said to be a **quaternionic Kähler manifold**. Every quaternionic Kähler manifold is orientable and of real dimension $4m$ for some $m \in \mathbb{N}_+$. For any $m > 1$, it is an Einstein manifold. As is usually done in quaternionic geometry, we therefore define a four-dimensional quaternionic Kähler manifold as a four-dimensional oriented self-dual Einstein space. A **quaternionic space form** of quaternionic sectional (Q-sectional) curvature $c \in \mathbb{R}$ is a connected quaternionic Kähler manifold $\bar{M}$ with the property that the Riemannian sectional curvature is equal to c for all tangent 2-planes spanned by $\{u, Ju\}$ with $u \in T_p\bar{M}$, $J \in \bar{\mathcal{J}}_p$, $p \in \bar{M}$, $\|u\| = 1$. The Riemannian curvature tensor $\bar{R}$ of a quaternionic space form $\bar{M}$ of Q-sectional curvature c is of the form

$$\begin{aligned}\bar{R}_{XY}Z =& \frac{c}{4}\Big[g(X,Z)Y - g(Y,Z)X \\ &+ \sum_{a=1}^{3}(g(J_aX,Z)J_aY - g(J_aY,Z)J_aX + 2g(J_aX,Y)J_aZ)\Big]\end{aligned} \tag{4.2}$$

for any canonical basis $\{J_1, J_2, J_3\}$ of $\bar{\mathcal{J}}$.

Now, let M be an orientable real hypersurface of a quaternionic Kähler manifold $\bar{M}$ and let N be a global unit normal field of M. Let p_r denote the canonical projection $T\bar{M}_{|M} \to TM$ and let $\mathcal{P}$ be the rank 3 vector subbundle of $\mathrm{Hom}(TM, T\bar{M})$ obtained as the restriction of $\bar{\mathcal{J}}$ to M. Then every canonical local basis $\{J_1, J_2, J_3\}$ of $\bar{\mathcal{J}}$ defined in an open neighborhood $\bar{\mathcal{U}}$ of $p \in M$ in $\bar{M}$ induces a basis $\{\varphi_1, \varphi_2, \varphi_3\}$ of $\mathcal{J} = p_r \circ \mathcal{P} \subset \mathrm{End}(TM)$ over $\mathcal{U} = \bar{\mathcal{U}} \cap M$ by defining $\varphi_a = p_r \circ J_{a|M}$. Each φ_a is a skew-symmetric $(1,1)$-tensor field on $\mathcal{U}$. Hence, for any vector field X on $\mathcal{U}$, we get

$$J_aX = \varphi_aX + \eta_a(X)N, \tag{4.3}$$

where η_a is the dual one-form of the vector field $\xi_a = -J_aN$ on $\mathcal{U}$ with respect to the metric g, $a = 1,2,3$. We also denote by g the induced Riemannian metric on M. We have $\eta_a(\xi_a) = 1$,

$$\varphi_a^2X = -X + \eta_a(X)\xi_a, g(\varphi_aX, \varphi_aY) = g(X,Y) - \eta_a(X)\eta_a(Y)$$

and also the following identities hold $(a \bmod 3)$:

$$\begin{gathered}\varphi_a\xi_a = 0, \quad \varphi_a\xi_{a+1} = \xi_{a+2}, \quad \varphi_a\xi_{a+2} = -\xi_{a+1}, \\ \varphi_a\varphi_{a+1} = \varphi_{a+2} = -\varphi_{a+1}\varphi_a.\end{gathered} \tag{4.4}$$

The Gauss and Weingarten formulas for M are given by

$$\bar{\nabla}_XY = \nabla_XY + g(SX,Y)N, \quad \bar{\nabla}_XN = -SX, \tag{4.5}$$

where $\bar{\nabla}$ and ∇ denote the Levi Civita connection of $(\bar{M}, g)$ and (M, g) respectively. Then, using (4.1) and (4.3), we get

$$\nabla_X\xi_a = q_{a+2}(X)\xi_{a+1} - q_{a+1}(X)\xi_{a+2} + \varphi_aSX \tag{4.6}$$

and

$$(\nabla_X\varphi_a)Y = q_{a+2}(X)\varphi_{a+1}Y - q_{a+1}(X)\varphi_{a+2}Y + \eta_a(Y)SX - g(SX,Y)\xi_a. \tag{4.7}$$

Furthermore, we have the following Gauss and Codazzi equations:

$$g(R_{XY}Z,W) = g(\bar{R}_{XY}Z,W) + g(SX,Z)g(SY,W) - g(SY,Z)g(SX,W), \tag{4.8}$$

$$g(\bar{R}_{XY}Z,N) = g((\nabla_Y S)X,Z) - g((\nabla_X S)Y,Z). \tag{4.9}$$

The three-dimensional oriented distribution $\mathcal{V}$ on M given by $\mathcal{V}_x = \{J_x N_x \mid J \in \bar{\mathcal{J}}\}$, $x \in M$, is called the **(quaternionic) Hopf distribution** on M and we shall denote by σ its associated section of $G_3^o(M)$. It is generated by the local (positive) orthonormal frame field $\{\xi_1, \xi_2, \xi_3\}$ and σ is locally expressed as $\sigma = \xi_1 \wedge \xi_2 \wedge \xi_3$. As in Section 3, we denote by $\{E_1, \ldots, E_n\}$, $n = 4m - 1$, a positive orthonormal local frame on M such that $E_a = \xi_a$, $a = 1, 2, 3$. Using (4.6), we have

$$\nabla_X \sigma = \sum_{a=1}^{3} \sum_{j=1}^{n-3} g(\varphi_a SX, E_{3+j})\sigma_j^a.$$

We start with the following criterion for the harmonicity of Hopf distributions when $\bar{M}$ is a general quaternionic Kähler manifold with $m > 1$.

THEOREM 4.1: *Let M be an orientable real hypersurface of a quaternionic Kähler manifold $\bar{M}$ with mean curvature h. Then the Hopf distribution is harmonic if and only if for each $a \in \{1, 2, 3\}$ the vector field*

$$\varphi_a(\operatorname{grad} h) + S^2\xi_a + 2\varphi_{a+1}S^2\xi_{a+2} - 2\varphi_{a+2}S^2\xi_{a+1}$$

is tangent to the distribution.

Proof: Using (4.6) and (4.7) for $a \in \{1, 2, 3\}$ and $j \in \{1, \ldots, n-3\}$, we have

$$\begin{aligned} g(\nabla^2_{YY}\xi_a, E_{3+j}) =& q_{a+2}(Y)g(\varphi_{a+1}SY, E_{3+j}) - q_{a+1}(Y)g(\varphi_{a+2}SY, E_{3+j}) \\ &+ g((\nabla_Y \varphi_a)SY, E_{3+j}) + g(\varphi_a(\nabla_Y S)Y, E_{3+j}) \\ =& 2q_{a+2}(Y)g(\varphi_{a+1}SY, E_{3+j}) - 2q_{a+1}(Y)g(\varphi_{a+2}SY, E_{3+j}) \\ &+ g(SY, \xi_a)g(SY, E_{3+j}) + g(\varphi_a(\nabla_Y S)Y, E_{3+j}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sum_{b=1}^{3} g(\nabla_Y \xi_b, \xi_a)g(\nabla_Y \xi_b, E_{3+j}) \\ &= g(\nabla_Y \xi_{a+1}, \xi_a)g(\nabla_Y \xi_{a+1}, E_{3+j}) + g(\nabla_Y \xi_{a+2}, \xi_a)g(\nabla_Y \xi_{a+2}, E_{3+j}) \\ &= q_{a+1}(Y)g(\varphi_{a+2}SY, E_{3+j}) - q_{a+2}(Y)g(\varphi_{a+1}SY, E_{3+j}) \\ &\quad - g(\xi_{a+1}, \varphi_a SY)g(\varphi_{a+1}SY, E_{3+j}) - g(\xi_{a+2}, \varphi_a SY)g(\varphi_{a+2}SY, E_{3+j}). \end{aligned}$$

Therefore, by (3.6), we get

$$\begin{aligned} g(\nabla^2_{YY}\sigma, \sigma^a_j) =& g(\varphi_a(\nabla_Y S)Y, E_{3+j}) + g(SY, \xi_a)g(SY, E_{3+j}) \\ &- 2g(\xi_{a+1}, \varphi_a SY)g(\varphi_{a+1}SY, E_{3+j}) \\ &- 2g(\xi_{a+2}, \varphi_a SY)g(\varphi_{a+2}SY, E_{3+j}). \end{aligned}$$

Since $\bar{M}$ is an Einstein manifold, we have from (4.9)

$$\sum_{\alpha=1}^{n}(\nabla_{E_\alpha}S)E_\alpha = \operatorname{grad} h.$$

Using the properties of the φ_a, we conclude that

$$\begin{aligned} g(\eta_{(\sigma,g)}, \sigma^a_j) &= \sum_{\alpha=1}^{n} g(\nabla^2_{E_\alpha E_\alpha}\sigma, \sigma^a_j) \\ &= g(\varphi_a(\operatorname{grad} h) + S^2\xi_a + 2\varphi_{a+1}S^2\xi_{a+2} - 2\varphi_{a+2}S^2\xi_{a+1}, E_{3+j}), \end{aligned}$$

from where the result follows. ∎

THEOREM 4.2: *Let M be an orientable real hypersurface of a quaternionic Kähler manifold $\bar{M}$. Then the Hopf distribution σ is minimal if and only if for each $a \in \{1,2,3\}$ the vector field*

$$\varphi_a(-\nabla^* SP) + SPS\xi_a + 2\varphi_{a+1}SPS\xi_{a+2} - 2\varphi_{a+2}SPS\xi_{a+1}$$

is tangent to the distribution. Here $P = L^{-1}_{\sigma^* g^S}$.

Proof: By Theorem 3.2(c), the minimality of σ is equivalent to the vanishing of $g(\eta_{(\sigma,\sigma^* g^S)}, \sigma^a_j)$ which, by Proposition 2.2, is equivalent to

$$-g(\nabla^* K_\sigma, \sigma^a_j) = \sum_{\alpha=1}^{n} g((\nabla_{E_\alpha}K_\sigma)E_\alpha, \sigma^a_j) = 0$$

where $K_\sigma = \nabla\sigma \circ P$ and ∇^* is the divergence operator. In a similar way as in the proof of Theorem 4.1, and by using (3.4), (4.6) and (4.7), we get

$$\begin{aligned} g((\nabla_Y K_\sigma)Y, \sigma^a_j) =& g(\varphi_a(\nabla_Y SP)Y, E_{3+j}) + g(\xi_a, SPY)g(SY, E_{3+j}) \\ &- g(\xi_{a+1}, \varphi_a SY)g(\varphi_{a+1}SPY, E_{3+j}) \\ &- g(\xi_{a+1}, \varphi_a SPY)g(\varphi_{a+1}SY, E_{3+j}) \\ &- g(\xi_{a+2}, \varphi_a SY)g(\varphi_{a+2}SPY, E_{3+j}) \\ &- g(\xi_{a+2}, \varphi_a SPY)g(\varphi_{a+2}SY, E_{3+j}). \end{aligned}$$

Therefore,

$$\sum_{\alpha=1}^{n} g((\nabla_{E_\alpha} K_\sigma)E_\alpha, \sigma_j^a) = g(\varphi_a(-\nabla^* SP) + SPS\xi_a, E_{3+j}) + g(2\varphi_{a+1} SPS\xi_{a+2} - 2\varphi_{a+2} SPS\xi_{a+1}, E_{3+j})$$

from where the result follows. ∎

Remark 4.3: From the proof of the theorem above, it is clear that if we consider any other metric $\tilde{g}$ on M and we put $P = L_{\tilde{g}}^{-1}$, the condition we obtain is then equivalent to σ being a critical point of the functional $E_{\tilde{g}}$.

Since $P = L_{\sigma^* g^S}^{-1} = \sqrt{(\det A)}A^{-1}$, where A is the endomorphism field defined by

$$\sigma^* g^S(X,Y) = g(AX,Y),$$

for a better understanding of the condition in Theorem 4.2, it will be convenient to write down the expression of the metric $\sigma^* g^S$. By arguments similar to those used in the proof of Proposition 3.10 for the characteristic distribution of a 3-Sasakian manifold, we easily get

LEMMA 4.4: *Let M be an orientable real hypersurface of a quaternionic Kähler manifold $\bar{M}$ and let σ be the Hopf distribution. Then*

$$AX = X + 3S^2X - 3\sum_{a=1}^{3} \eta_a(SX)S\xi_a.$$

The extrinsic geometry of the submanifold M is so involved in the criticality conditions of the Hopf distribution that it is difficult to go further, without making assumptions on the behaviour of the shape operator. The conditions in Theorem 4.1 and Theorem 4.2 simplify a great deal when we restrict our attention to the well-known class of the Hopf hypersurfaces.

If $\mathcal{V}$ (or equivalently, the orthogonal distribution $\mathcal{H}$) is invariant by S, then M is called a **Hopf hypersurface** of $\bar{M}$. From (4.6), the Hopf distribution $\mathcal{V}$ is invariant by the shape operator S of M if and only if it is autoparallel, that is, $\nabla_X Y \in \mathcal{V}$ for all $X, Y \in \mathcal{V}$ (see also Proposition 3.5 of [1]). In particular, $\mathcal{V}$ is integrable and the induced foliation $\mathcal{F}_\mathcal{V}$ is totally geodesic. If M is a Hopf hypersurface, then there exists an open and dense subset W of M where the eigenvalues of S are smooth and their multiplicities are locally constant. It yields that for each $p \in W$ there exists a canonical local basis $\{J_1, J_2, J_3\}$ of $\bar{\mathcal{J}}$ such that ξ_a is an eigenvector of S, defined on some open $\bar{\mathcal{U}}$ with $p \in \mathcal{U} = \bar{\mathcal{U}} \cap M \subset W$ (see [1, Lemma 3.6]).

This local frame of $\mathcal{V}$ can be completed to an orthonormal frame field $\{E_a = \xi_a; E_{3+j}\}$ on $\mathcal{U}$ satisfying $SE_\alpha = \lambda_\alpha E_\alpha$, $\alpha = 1, \dots, n$. Moreover, from Lemma 4.4, we have $P(E_a) = f(\sigma)E_a$ and $P(E_{3+j}) = (f(\sigma)/(1 + 3\lambda_{3+j}^2))E_{3+j}$ where

$$f(\sigma)^2 = \prod_{j=1}^{n-3} (1 + 3\lambda_{3+j}^2).$$

THEOREM 4.5: *Let M be a Hopf hypersurface of a quaternionic Kähler manifold $\bar{M}$.*

(a) *The Hopf distribution is harmonic if and only if the gradient of the mean curvature of M is tangent to the distribution. In particular, if M has constant mean curvature, then the Hopf distribution of M is harmonic.*

(b) *If the Hopf distribution is harmonic, then it determines a harmonic map from (M, g) to $(G_3^o(M), g^S)$ if and only if, for any local orthonormal frame of $\mathcal{H}$ comprising eigenvectors of the shape operator, the vector field*

$$\sum_{k=4}^{n} \lambda_k \bar{R}(N, E_k)E_k$$

is normal to M.

(c) *The Hopf distribution gives a minimal immersion into $(G_3^o(M), g^S)$ if and only if $\nabla^* SP$ is tangent to the distribution. This condition can be expressed as*

$$\sum_{k=4}^{n} \frac{1}{1 + 3\lambda_k^2}(E_j(\lambda_k) + (1 - 3\lambda_k\lambda_j)\bar{R}(N, E_k, E_k, E_j)) = 0$$

for all $j = 4, \dots, n$.

Proof: Part (a) is a direct consequence of Theorem 4.1. Now we prove (b).

By Theorem 3.2, a harmonic distribution σ defines a harmonic map if and only if, for each vector field X on M,

$$\sum_{\alpha=1}^{n} g(R(X, E_\alpha)\sigma, \nabla_{E_\alpha}\sigma) = 0.$$

Using (3.8) and the fact that by hypothesis $S\xi_a \in \mathcal{V}$ and $SE_j \in \mathcal{H}$, we have

$$\sum_{\alpha=1}^{n} g(R(X,E_\alpha)\sigma, \nabla_{E_\alpha}\sigma) = \sum_{a=1}^{3}\sum_{j,k=4}^{n} g(\varphi_a SE_k, E_j) g(R(X,E_k)\xi_a, E_j)$$
$$= \sum_{a=1}^{3}\sum_{k=4}^{n} g(R(X,E_k)\xi_a, \varphi_a SE_k).$$

Now, from the Gauss equation (4.8) and the properties of the curvature tensor on a quaternionic Kähler manifold [4, p. 403], we derive

$$\begin{aligned} g(R(X,E_k)\xi_a, \varphi_a SE_k) &= g(\bar{R}(X,E_k)\xi_a, \varphi_a SE_k) \\ &= -g(\bar{R}(X,E_k)J_aN, J_aSE_k) \\ &= -g(\bar{R}(X,E_k)N, SE_k). \end{aligned}$$

Therefore,

$$\sum_{\alpha=1}^{n} g(R(X,E_\alpha)\sigma, \nabla_{E_\alpha}\sigma) = -3\sum_{k=4}^{n}(\bar{R}(X,E_k)N, SE_k),$$

from where (b) holds.

The first assertion of (c) is a consequence of Theorem 4.2 and the definition of a Hopf hypersurface. Furthermore, by Lemma 4.4, $SP(E_\alpha) = \tilde{\lambda}_\alpha$ with $\tilde{\lambda}_a = f(\sigma)\lambda_a$ and $\tilde{\lambda}_j = f(\sigma)\lambda_j/(1+3\lambda_j^2)$. Therefore

$$g(\nabla^* SP, E_j) = -E_j(\tilde{\lambda}_j) + \sum_{\alpha=1}^{n}(\tilde{\lambda}_j - \tilde{\lambda}_\alpha) g(\nabla_{E_\alpha}E_\alpha, E_j).$$

Since $\nabla_{\xi_a}\xi_a = 0$ and

$$E_j(\tilde{\lambda}_j) = \frac{f(\sigma)}{(1+3\lambda_j^2)^2}E_j(\lambda_j) + \frac{f(\sigma)\lambda_j}{1+3\lambda_j^2}\sum_{k\neq j=4}^{n}\frac{\lambda_k}{1+3\lambda_k^2}E_j(\lambda_k),$$

the vector field $\nabla^* SP$ is tangent to the Hopf distribution if and only if for all $j = 4,\ldots,n$

$$\frac{1}{1+3\lambda_j^2}E_j(\lambda_j) + \sum_{k\neq j=4}^{n}\Big(\frac{3\lambda_j\lambda_k}{1+3\lambda_k^2}E_j(\lambda_k) + \frac{(\lambda_k-\lambda_j)(1-3\lambda_k\lambda_j)}{1+3\lambda_k^2} g(\nabla_{E_k}E_k, E_j)\Big) = 0.$$

Now, for $k \neq j$, the Codazzi equation reads as

$$g(\bar{R}(E_j,E_k)E_k, N) = -E_j(\lambda_k) + (\lambda_j - \lambda_k) g(\nabla_{E_k}E_j, E_k). \tag{4.10}$$

Therefore, the condition above is equivalent to

$$\sum_{k=4}^{n} \frac{1}{1+3\lambda_k^2} E_j(\lambda_k) + \sum_{k=4}^{n} \frac{1-3\lambda_k\lambda_j}{1+3\lambda_k^2} g(\bar{R}(E_j, E_k)E_k, N) = 0,$$

and so we have shown the second assertion. ∎

Let $G_2(\mathbb{C}^{m+2})$ be the complex Grassmannian manifold of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Denote by g its canonical metric. Then $(G_2(\mathbb{C}^{m+2}), g)$ is a compact Hermitian symmetric space and this manifold and its non-compact dual $G_2(\mathbb{C}^{m+2})^*$ are Einstein spaces. These symmetric spaces are equipped with a Kähler structure J and a quaternionic Kähler structure $\mathcal{J}$. In [2], for $m \geq 3$, a complete classification is given for those real hypersurfaces of $G_2(\mathbb{C}^{m+2})$ and of $G_2(\mathbb{C}^{m+2})^*$ which are Hopf hypersurfaces for both structures. Such hypersurfaces are tubes about special totally geodesic submanifolds with constant principal curvatures. Moreover, they satisfy that, for any unit principal curvature vector E, $\bar{R}(N,E)E$ is normal to the hypersurface [23]. Hence, we have

COROLLARY 4.6: *The Hopf distribution induced by the quaternionic Kähler structure $\mathcal{J}$ of $G_2(\mathbb{C}^{m+2})$ and of $G_2(\mathbb{C}^{m+2})^*$ on each one of such real hypersurfaces M determines a harmonic map of (M,g) into $(G_3^o(M), g^S)$ and a minimal immersion of $(M, \sigma^* g^S)$ into $(G_3^o(M), g^S)$.*

Finally, we consider the particular case where M is a Hopf hypersurface of a quaternionic space form $\bar{M}$ of Q-sectional curvature c. Using the expression of the curvature tensor given in (4.2), we see that for any vector field E tangent to the hypersurface, $\bar{R}(N,E)E$ is normal to M. Let E be a local unit vector field of $\mathcal{H}$ such that $SE = \lambda E$. Then for each $a \in \{1,2,3\}$, using (4.10), we have

$$0 = \xi_a(\lambda) + (\lambda - \lambda_a) g(\nabla_E \xi_a, E),$$
$$0 = E(\lambda_a) + (\lambda_a - \lambda) g(\nabla_{\xi_a} E, \xi_a).$$

Hence and from (4.6) it follows that $\xi_a(\lambda) = 0$ and $E(\lambda_a) = 0$. Moreover, if the Q-sectional curvature of $\bar{M}$ is $c \neq 0$, and according to [1, Lemma 4.16], each of the principal curvatures λ_a is constant on $\mathcal{U}$.

Therefore we obtain the following Corollary of Theorem 4.5.

COROLLARY 4.7: *Let $\mathcal{V}$ be the Hopf distribution on a Hopf hypersurface M of a quaternionic space form $\bar{M}$ of Q-sectional curvature c. Then we have:*

(a) *$\mathcal{V}$ is harmonic if and only if $\sum_{k=4}^{n} \lambda_k$ is constant;*

(b) $\mathcal{V}$ *determines a harmonic map of* (M, g) *into* $(G_3^o(M), g^S)$ *if and only if it is harmonic;*

(c) $\mathcal{V}$ *determines a minimal immersion of* $(M, \sigma^* g^S)$ *into* $(G_3^o(M), g^S)$ *if and only if* $\sum_{k=4}^n \arctan(\sqrt{3}\lambda_k)$ *is constant.*

Moreover, if $c \neq 0$ *then* $\mathcal{V}$ *is harmonic if and only if* M *has constant mean curvature.*

Proof: The only part that needs some hint is (a). We know that $\mathcal{V}$ is harmonic if and only if $E_j(h) = 0$ for all $j = 4, \ldots, n$. Under the hypothesis,

$$E_j(h) = E_j\left(\sum_{k=4}^{n} \lambda_k\right),$$

and since $\xi_a(\lambda_k) = 0$, we get the result. ∎

It has been shown in [1] that for the quaternionic projective space $\mathbb{H}P^m(c)$, $m \geq 2$, of constant quaternionic sectional curvature $c > 0$, the principal curvatures of a Hopf hypersurface are locally constant and the classification of these Hopf hypersurfaces is also given there. Moreover, one can also find in [1] the classification of Hopf hypersurfaces with constant principal curvatures in the quaternionic hyperbolic space $\mathbb{H}H^m(-c)$, $m \geq 2$. Then we can state

COROLLARY 4.8: *The Hopf distribution on a Hopf hypersurface in the quaternionic projective space* $\mathbb{H}P^m(c)$, $m \geq 2$, *determines a harmonic map and a minimal immersion. The same result holds for Hopf hypersurfaces with constant principal curvatures in* $\mathbb{H}H^m(-c)$, $m \geq 2$.

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